A stochastic process driven by the quadratic Ornstein-Uhlenbeck noise: generator, propagators and all that

1988 J. Phys. A: Math. Gen. 21 3063

(http://iopscience.iop.org/0305-4470/21/14/008)

J. Phys. A: Math. Gen. 21 (1988) 3063-3077. Printed in the UK

A stochastic process driven by the quadratic Ornstein–Uhlenbeck noise: generator, propagators and all that

J Łuczka

Department of Theoretical Physics, Silesian University, 40-007 Katowice, Poland

Received 13 January 1988

Abstract. A non-Markovian stochastic process modelled by a linear first-order differential equation involving quadratic Ornstein-Uhlenbeck noise is investigated. The generator of an evolution operator of the process is constructed and linear propagators of a one-dimensional probability distribution are built. The initial correlation functions are presented and evolution equations for the moments of the process are derived. Some approximative methods are verified.

1. Introduction

Evolution equations with random parameters describe a great variety of physical [1], chemical [2] and biological [3] systems. In many cases, they are ordinary differential equations and, in general, these equations are the result of a reduction of more fundamental equations describing the microscopic level of the system. If characteristics of random parameters are known then the evolution of the system is described by stochastic equations of the form

$$\dot{x}_t = f(x_t, y_t) \tag{1.1}$$

with random parameters modelled by the stochastic process y_t. In the general case, random parameters enter non-linearly and multiplicatively in the deterministic equation. This situation is referred to as non-linear noise. Equation (1.1) defines a stochastic process x_t and the main properties of this process can be described by a single-event probability distribution $p(x, t)$. One of the fundamental problems for analysis of (1.1) is to construct an infinitesimal generator of the evolution operator of the stochastic process x_t. This would allow us, for example, to find an evolution equation of the probability distribution $p(x, t)$ of the process x_t. Unfortunately, this problem is seldom amenable to exact analytic solution so there are many approximation procedures which have been proposed [4–6]. Therefore, it is worthwhile studying the stochastic equations for which this problem can be exactly solved.

In this paper, we study a simple equation

$$\dot{x}_t = f(y_t)x_t \tag{1.2}$$

where f is a quadratic function of the noise y_t.

There are several reasons for studying (1.2). Firstly, this equation describes many physical phenomena (see [7, 8] and references therein). For example, the relaxation equation

$$\dot{x}_t = -E^2 x_t \tag{1.3}$$

under the assumption that the parameter E may fluctuate:

$$E = E_0 + y_t \tag{1.4}$$

where E_0 is a constant, also belongs to the class of equations given by (1.2). Secondly, equation (1.2) can be exactly solved in the sense that whole one-dimensional dynamics (a generator, propagators, $p(x, t)$) are given in analytical form.

Because of the non-linearity in y_t, the noise cannot be treated as white noise. Here we assume that y_t is coloured noise modelled by an Ornstein–Uhlenbeck stochastic process [9]. Recently [10], we have studied a linear differential equation with an additive quadratic noise. It is worthwhile comparing these two models.

The linear differential equations with multiplicative noise as (1.2) have been studied in the literature. San Miguel and Sancho [6] derived an approximative evolution equation for $p(x, t)$. The exact mean value of x_t was obtained in [7, 8]. Wódkiewicz [11] derived the exact evolution equation for $p(x, t)$. His equation is a partial integro-differential equation with retardation (memory). The kernel of the integral part of this equation is the inverse Laplace transform of the operator-continued fraction. The theory of such a type of equation was not elaborated. Therefore it is difficult to analyse such equations. In this paper we present a more convenient evolution equation for $p(x, t)$.

In § 2 we formulate the problem of interest. It is desirable to transform the starting process x_t into another process z_t. In § 3 we derive an exact evolution equation for the probability distribution $P(z, t)$ of the transformed process z_t. This equation is obtained directly from the Fokker–Planck equation for the joint probability density of a two-dimensional diffusion process (z_t, y_t). By solving the suitable Martin–Siggia–Rose equations [12], we are able to eliminate the noise variable from the Fokker–Planck equation. It is a non-trivial example of obtaining the reduced equation for the 'slow' variable z by using the procedure of elimination of the 'fast' variable y applied to partial differential equations [13]. In § 4 we present a master-type equation for the density $p(x, t)$ of the starting process x_t. In § 5 we construct the infinitesimal generator of the evolution operator for x_t. This generator has the form of the Kramers–Moyal operator [14] with explicitly defined coefficients. In § 6 we solve the master-type equation to obtain $p(x, t)$ and we construct propagators. In § 7 we present the initial correlation functions which in special cases correspond to the mean value and fluctuations of the process x_t. The evolution equations for the moments of x_t are also contained in § 7. In § 8 we discuss some approximative methods considered in the literature.

2. Formulation of the problem

The equation we are considering is [6–8]

$$\dot{x}_t = -(A + By_t + Cy_t^2)x_t \qquad x \in \mathbb{R}^+ \tag{2.1}$$

where A, B and C are fixed constants. The coloured noise y_t is assumed to be the Ornstein–Uhlenbeck process [9] of mean value zero and correlation

$$\langle y_t y_s \rangle = (\gamma/\alpha) \exp(-\alpha|t-s|) \tag{2.2}$$

with γ and α fixed positive constants.

This process is generated by the stochastic differential

$$\mathrm{d}y_t = -\alpha y_t\,\mathrm{d}t + (2\gamma)^{1/2}\,\mathrm{d}W_t \qquad y \in \mathbb{R} \tag{2.3}$$

under the assumption

$$\langle y_0 \rangle = 0 \qquad \langle y_0^2 \rangle = \gamma/\alpha. \tag{2.4}$$

Here, $\langle\ \rangle$ denotes the expectation value of the process and W_t is the standard Wiener process.

The probability density $\mathcal{P}_0(y, t)$ of y_t has the Gaussian form

$$\mathcal{P}_0(y, t) = \mathcal{P}_0(y) = (\alpha/2\pi\gamma)^{1/2} \exp(-\alpha y^2/2\gamma). \tag{2.5}$$

We assume that the initial probability distribution $p(x, 0)$ of x_t in (2.1) is given by

$$p(x, 0) = p(x) \tag{2.6}$$

and x_0 is statistically independent of the noise y_t. If, e.g., in (2.1) x_0 is fixed, $x_0 = c$, then

$$p(x) = \delta(x - c). \tag{2.7}$$

If the problem (2.1) is solved then the corresponding problem with $x \in \mathbb{R}^-$ can be easily solved by noting that (2.1) is invariant with respect to the change of sign of x. The normalisation condition for $p(x, t)$ has the form

$$\int_0^\infty p(x, t)\,\mathrm{d}x = 1. \tag{2.8}$$

Let us introduce a new variable

$$z = \ln x. \tag{2.9}$$

Then (2.1) takes the form

$$\dot{z}_t = -(A + By_t + Cy_t^2) \qquad z \in \mathbb{R} \tag{2.10}$$

and the initial distribution $P(z, 0)$ of z_t is given by

$$P(z, 0) = \mathrm{e}^z p(\mathrm{e}^z). \tag{2.11}$$

The distribution $p(x, t)$ can be obtained from $P(z, t)$ of z_t in (2.10) via the relation

$$p(x, t) = \frac{1}{x} P(z = \ln x, t). \tag{2.12}$$

First, we find an evolution equation for $P(z, t)$ and then we transform it to obtain an equation for $p(x, t)$.

3. Master-type equation for $P(z, t)$

Let us notice that the two-dimensional stochastic process (z_t, y_t) is a degenerate diffusion process and the joint probability density $\rho(z, y, t)$ obeys the Fokker–Planck equation

$$\partial\rho(z, y, t)/\partial t = L\rho(z, y, t) \tag{3.1}$$

where the Fokker–Planck generator L has the form

$$L = (A + By + Cy^2)\frac{\partial}{\partial z} + \alpha\frac{\partial}{\partial y}y + \gamma\frac{\partial^2}{\partial y^2} \tag{3.2}$$

and the Cauchy boundary condition for (3.1) has the form

$$\rho(z, y, 0) = \mathscr{P}_0(y) P(z, 0). \tag{3.3}$$

The formal solution of (3.1) is

$$\rho(z, y, t) = \mathrm{e}^{Lt} \rho(z, y, 0) \tag{3.4}$$

and the probability distribution $P(z, t)$ can be obtained from

$$P(z, t) = \int_{-\infty}^{\infty} \mathrm{d}y \, \mathrm{e}^{Lt} \mathscr{P}_0(y) P(z, 0) = U(t) P(z, 0) \tag{3.5}$$

where the evolution operator $U(t)$ of the process z_t has the form

$$U(t) = \int_{-\infty}^{\infty} \mathrm{d}y \, \mathrm{e}^{Lt} \mathscr{P}_0(y). \tag{3.6}$$

For notational convenience we have dropped the dependence of $U(t)$ on the differential operator $\partial/\partial z$. The infinitesimal generator $l(t)$ of z_t is determined by the relation

$$\dot{U}(t) = l(t) U(t) \tag{3.7a}$$

or by the equation

$$\frac{\partial P(z, t)}{\partial t} = l(t) P(z, t). \tag{3.7b}$$

In the case considered, the generator $l(t)$ can be explicitly evaluated. Let us notice that

$$\begin{aligned} \dot{U}(t) &= \int_{-\infty}^{\infty} \mathrm{d}y \, L \, \mathrm{e}^{Lt} \mathscr{P}_0(y) \\ &= \frac{\partial}{\partial z} \int_{-\infty}^{\infty} \mathrm{d}y (A + By + Cy^2) \, \mathrm{e}^{Lt} \mathscr{P}_0(y) \\ &= A \frac{\partial}{\partial z} U(t) + B \frac{\partial}{\partial z} V(t) + C \frac{\partial}{\partial z} W(t) \end{aligned} \tag{3.8}$$

where

$$V(t) = \int_{-\infty}^{\infty} \mathrm{d}y \, y \, \mathrm{e}^{Lt} \mathscr{P}_0(y) = \int_{-\infty}^{\infty} \mathrm{d}y \, \mathrm{e}^{Lt} y(t) \mathscr{P}_0(y) \tag{3.9}$$

and

$$W(t) = \int_{-\infty}^{\infty} \mathrm{d}y \, y^2 \, \mathrm{e}^{Lt} \mathscr{P}_0(y) = \int_{-\infty}^{\infty} \mathrm{d}y \, y \, \mathrm{e}^{Lt} y(t) \mathscr{P}_0(y) \tag{3.10}$$

where

$$y(t) = \mathrm{e}^{-Lt} y \, \mathrm{e}^{Lt}. \tag{3.11}$$

Now, we will express the operators $V(t)$ and $W(t)$ in terms of $U(t)$. The procedure is similar to that used in [10].

3.1. *Martin–Siggia–Rose equations*

To find the time dependence of $y(t)$ given by (3.11), we should solve suitable Martin–Siggia–Rose equations [12]:

$$\dot{y}(t) = -[L, y(t)] = -\alpha y(t) - 2\gamma\hat{y}(t) \tag{3.12a}$$

$$\dot{\hat{y}}(t) = -[L, \hat{y}(t)] = \alpha\hat{y}(t) + B\hat{z}(t) + 2C\hat{z}(t)y(t) \tag{3.12b}$$

$$\dot{\hat{z}}(t) = -[L, \hat{z}(t)] = 0 \tag{3.12c}$$

with the initial conditions

$$y(0) = y \qquad \hat{y}(0) = \hat{y} \qquad \hat{z}(0) = \hat{z} \tag{3.13}$$

and for notational convenience we have introduced

$$\hat{y} = \frac{\partial}{\partial y} \qquad \hat{z} = \frac{\partial}{\partial z}. \tag{3.14}$$

The time dependence of $\hat{y}(t)$ and $\hat{z}(t)$ are given by similar equations to (3.11). From (3.12c) it follows that

$$\hat{z}(t) = \hat{z} \tag{3.15}$$

and

$$[\hat{z}(t), y(t)] = [\hat{z}(t), \hat{y}(t)] = 0. \tag{3.16}$$

The system of equations (3.12) can be solved by the standard method. The solution of (3.12) has the form

$$y(t) = f_1(t, \hat{z})y + f_2(t, \hat{z})\hat{y} + f_3(t, \hat{z}) \tag{3.17a}$$

$$\hat{y}(t) = h_1(t, \hat{z})\hat{y} + h_2(t, \hat{z})y + h_3(t, \hat{z}) \tag{3.17b}$$

where

$$f_1(t, \hat{z}) = \cosh\omega(\hat{z})t - \alpha\frac{\sinh\omega(\hat{z})t}{\omega(\hat{z})} \tag{3.18a}$$

$$f_2(t, \hat{z}) = -2\gamma\frac{\sinh\omega(\hat{z})t}{\omega(\hat{z})} \tag{3.18b}$$

$$f_3(t, \hat{z}) = -2\gamma B\hat{z}\frac{\cosh\omega(\hat{z})t - 1}{\omega^2(\hat{z})} \tag{3.18c}$$

and

$$h_1(t, \hat{z}) = \cosh\omega(\hat{z})t + \alpha\frac{\sinh\omega(\hat{z})t}{\omega(\hat{z})} \tag{3.19a}$$

$$h_2(t, \hat{z}) = 2C\hat{z}\frac{\sinh\omega(\hat{z})t}{\omega(\hat{z})} \tag{3.19b}$$

$$h_3(t, \hat{z}) = B\hat{z}\left(\alpha\frac{\cosh\omega(\hat{z})t - 1}{\omega^2(\hat{z})} + \frac{\sinh\omega(\hat{z})t}{\omega(\hat{z})}\right) \tag{3.19c}$$

and the 'frequency' ω is defined by the formula

$$\omega^2(\hat{z}) = \alpha^2 - 4\gamma C\hat{z}. \tag{3.20}$$

The operator-valued functions in (3.18) and (3.19) are defined by the power series of the appropriate elementary functions.

3.2. *Relation between $V(t)$ and $U(t)$*

Inserting (3.17a) into (3.9) and using the relation (cf (2.5))

$$\gamma\hat{y}\mathscr{P}_0(y) = -\alpha y\mathscr{P}_0(y) \tag{3.21}$$

we can rewrite equation (3.9) as follows

$$V(t) = f_3(t, \hat{z})\,U(t) + h_1(t, \hat{z}) \int_{-\infty}^{\infty} \mathrm{d}y\, y(-t)\, \mathrm{e}^{L't}\mathscr{P}_0(y) \tag{3.22a}$$

or in the form

$$V(t) = f_3(t, \hat{z})\,U(t) - \frac{\gamma}{\alpha}\, h_1(t, \hat{z}) \int_{-\infty}^{\infty} \mathrm{d}y\, \hat{y}(-t)\, \mathrm{e}^{L't}\mathscr{P}_0(y). \tag{3.22b}$$

By using once more the solution (3.17) for $y(-t)$, $\hat{y}(-t)$ and equations (3.22), we obtain the relation between $V(t)$ and $U(t)$:

$$Q(t, \hat{z})\,V(t) = \gamma h_3(t, \hat{z})\,U(t) \tag{3.23}$$

where

$$Q(t, \hat{z}) = \alpha \cosh \omega(\hat{z})t + (\alpha^2 - 2\gamma C\hat{z})\,\frac{\sinh \omega(\hat{z})t}{\omega(\hat{z})}. \tag{3.24}$$

To obtain (3.23), one should use the same arguments as in appendix C in [10].

3.3. *Relation between $W(t)$ and $U(t)$*

Applying the same techniques as in § 3.2, we are able to show that from (3.10) it follows that

$$Q(t, \hat{z})\,W(t) = \gamma h_1(t, \hat{z})\,U(t) + \gamma h_3(t, \hat{z})\,V(t). \tag{3.25}$$

Applying the operator $Q(t, \hat{z})$ on both sides of equation (3.25), and using (3.23), we get

$$Q^2(t, \hat{z})\,W(t) = [\gamma Q(t, \hat{z})h_1(t, \hat{z}) + \gamma^2 h_3^2(t, \hat{z})]\,U(t). \tag{3.26}$$

Now, the problem is to find the inverse of $Q(t, \hat{z})$ and $Q^2(t, \hat{z})$ which are differential operators of infinite order.

3.4. *Inverse operators*

The operators $U(t)$, $V(t)$, $W(t)$ and $Q(t, \hat{z})$ act in the proper space of distributions. Let g be any element of this space. If we denote

$$V(t)g(z) = g_1(z, t) \tag{3.27a}$$

$$\gamma h_3(t, \hat{z})\,U(t)g(z) = g_2(z, t) \tag{3.27b}$$

then (3.23) takes the form

$$Q(t, \hat{z})g_1(z, t) = g_2(z, t). \tag{3.28}$$

This is an ordinary differential equation of infinite order. There is a rich bibliography on such equations and the reader is referred to the paper by Leontiev [15]. We treat

(3.28) in the same way as equations of finite order. Using the Fourier transformation method, from (3.28), we find

$$g_1(z, t) = \int_{-\infty}^{\infty} \mathrm{d}z' \, H(z-z', t) g_2(z', t) \tag{3.29}$$

where

$$H(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k \frac{\mathrm{e}^{-\mathrm{i}kz}}{Q(t, -\mathrm{i}k)}. \tag{3.30}$$

Using now (3.27) and the properties of the Fourier transforms, we obtain

$$V(t) g(z) = \int_{-\infty}^{\infty} \mathrm{d}z' \, G_1(z-z', t) U(t) g(z') \tag{3.31}$$

where

$$G_n(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k \, \mathrm{e}^{-\mathrm{i}kz} \hat{G}_n(k, t) \qquad n = 1, 2 \tag{3.32}$$

with

$$\hat{G}_1(k, t) = \gamma \frac{h_3(t, -\mathrm{i}k)}{Q(t, -\mathrm{i}k)}. \tag{3.33}$$

In the same way, from (3.26), we obtain

$$W(t) g(z) = \int_{-\infty}^{\infty} \mathrm{d}z' \, G_2(z-z', t) U(t) g(z') \tag{3.34}$$

and the Fourier transform $\hat{G}_2(k, t)$ in (3.32) has the form

$$\hat{G}_2(k, t) = \gamma \frac{h_1(t, -\mathrm{i}k)}{Q(t, -\mathrm{i}k)} + \gamma^2 \frac{h_3^2(t, -\mathrm{i}k)}{Q^2(t, -\mathrm{i}k)}. \tag{3.35}$$

3.5. Evolution equation

From equation (3.5) it follows that

$$\frac{\partial P(z, t)}{\partial t} = \dot{U}(t) P(z, 0). \tag{3.36}$$

Using equations (3.8), (3.31) and (3.34) we obtain the desired equation for the time evolution of $P(z, t)$. It has the form

$$\frac{\partial P(z, t)}{\partial t} = A \frac{\partial}{\partial z} P(z, t) + \frac{\partial}{\partial z} \int_{-\infty}^{\infty} \mathrm{d}z' \, G(z-z', t) P(z', t) \tag{3.37}$$

where

$$G(z, t) = B G_1(z, t) + C G_2(z, t). \tag{3.38}$$

4. Master-type equation for $p(x, t)$

We can use the transformation (2.9) and the relation (2.12) to convert (3.37) into an evolution equation for the probability distribution $p(x, t)$ of the starting process x_t. It takes the form

$$\frac{\partial p(x, t)}{\partial t} = A\frac{\partial}{\partial x}[xp(x, t)] + \frac{\partial}{\partial x}\int_0^\infty \mathrm{d}x'\, G\left(\ln\frac{x}{x'}, t\right)p(x', t). \tag{4.1}$$

This equation can be recast in the master-type equation [16]

$$\frac{\partial p(x, t)}{\partial t} = \int_{-\infty}^{\infty} \mathrm{d}x'\, \mathscr{W}_t(x|x')p(x', t) \tag{4.2}$$

where

$$\mathscr{W}_t(x|x') = \left[A\delta(x-x') + Ax\frac{\partial}{\partial x}\delta(x-x') + \frac{\partial}{\partial x}G\left(\ln\frac{x}{x'}, t\right)\right]\theta(x') \tag{4.3}$$

and $\theta(x)$ denotes the Heaviside function. Equation (4.2) is the first important result of this paper. We call it a master-type equation since (4.2) has a similar structure to the master equation for a one-dimensional probability distribution (or a conditional distribution) of Markovian processes [16]. The equation derived by Wódkiewicz [11] is an integro-differential equation of the time-convolution type (an equation with memory) whereas our equation (4.2) is time convolutionless.

5. Generator of the process x_t

The generator of the time translation of the process x_t in (2.1) is determined by equation (4.2). In this section we derive another form of this generator. To do this, it is useful to rewrite (3.37) in the form

$$\frac{\partial P(z, t)}{\partial t} = A\frac{\partial}{\partial z}P(z, t) + \frac{\partial}{\partial z}\int_{-\infty}^{\infty} \mathrm{d}z'\, G(z', t)\exp\left(-z'\frac{\partial}{\partial z}\right)P(z, t) \tag{5.1}$$

where we have utilised the property of commutativity of convolution in (3.37) and introduced the shift operator

$$T_{z'}P(z, t) = P(z-z', t) = \exp\left(-z'\frac{\partial}{\partial z}\right)P(z, t). \tag{5.2}$$

By use of the transformation (2.9) and the relation (2.12), from (5.1) we obtain another form of the evolution equation for $p(x, t)$, namely

$$\frac{\partial p(x, t)}{\partial t} = A\frac{\partial}{\partial x}[xp(x, t)] + \frac{\partial}{\partial x}\int_{-\infty}^{\infty} \mathrm{d}z\, G(z, t)\exp\left(-zx\frac{\partial}{\partial x}\right)[xp(x, t)]. \tag{5.3}$$

Using the operator identity (see the appendix)

$$\exp\left(-zx\frac{\partial}{\partial x}\right) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\mathrm{e}^z(\mathrm{e}^z-1)^n\frac{\partial^n}{\partial x^n}x^n \tag{5.4}$$

we can write equation (5.3) in the form

$$\frac{\partial p(x, t)}{\partial t} = \mathscr{L}(t)p(x, t) \tag{5.5}$$

where the generator $\mathscr{L}(t)$ of x_t is the Kramers–Moyal operator [14]

$$\mathscr{L}(t)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{\partial^n}{\partial x^n}M_n(x,t) \tag{5.6}$$

where

$$M_n(x,t)=-n[A\delta_{n,1}+K_n(t)]x^n \tag{5.7}$$

and

$$\begin{aligned}K_n(t)&=\int_{-\infty}^{\infty}\mathrm{d}z\,\mathrm{e}^z(\mathrm{e}^z-1)^{n-1}G(z,t)\\&=\sum_{m=0}^{n-1}(-1)^{n-m-1}\binom{n-1}{m}\hat{G}(-\mathrm{i}(m+1),t).\end{aligned} \tag{5.8}$$

By virtue of equations (3.19), (3.24), (3.33), (3.35) and (3.38) all functions $\hat{G}(-\mathrm{i}(m+1), t)$ are explicitly determined (in (3.20), one should put $\omega^2(-(m+1))=\alpha^2+4(m+1)\gamma C$).

6. Propagators of the process x_t

The solution of the Kramers–Moyal equation (5.6) or the master-type equation (4.2) can be written in terms of propagators $\Pi(x,t|x',0)$ as follows:

$$p(x,t)=\int_0^{\infty}\mathrm{d}x'\,\Pi(x,t|x',0)p(x',0) \tag{6.1}$$

with a given initial distribution density $p(x,0)$. To construct the propagators, we first solve (3.37). This equation can be solved by the Fourier transformation method. From (3.37) it follows that the characteristic function

$$C^z(k,t)=\int_{-\infty}^{\infty}\mathrm{d}z\,\mathrm{e}^{\mathrm{i}kz}P(z,t) \tag{6.2}$$

of the transformed process z_t obeys the following equation:

$$\frac{\partial C^z(k,t)}{\partial t}=-\mathrm{i}k[A+\hat{G}(k,t)]C^z(k,t) \tag{6.3}$$

with the initial condition $C^z(k,0)$ which follows from (6.2). The solution of equation (6.3) has the form

$$C^z(k,t)=C^z(k,0)\hat{\mathbb{F}}(k,t) \tag{6.4}$$

$$\hat{\mathbb{F}}(k,t)=\exp\left(-\mathrm{i}kAt-\mathrm{i}k\int_0^t\hat{G}(k,s)\,\mathrm{d}s\right). \tag{6.5}$$

The integral in (6.5) can be evaluated [17] and we obtain

$$\begin{aligned}\hat{\mathbb{F}}(k,t)=&\left(\cosh\alpha\Omega t+\frac{\Omega^2+1}{2\Omega}\sinh\alpha\Omega t\right)^{-1/2}\\&\times\exp\left(\tfrac{1}{2}\alpha t-\mathrm{i}kAt-\frac{k^2B^2\gamma}{\alpha^2\Omega^2}t\right)\\&\times\exp\left(\frac{2k^2B^2\gamma}{\alpha^3\Omega^3}\frac{\Omega\sinh\alpha\Omega t+\cosh\alpha\Omega t-1}{2\Omega\cosh\alpha\Omega t+(\Omega^2+1)\sinh\alpha\Omega t}\right)\end{aligned} \tag{6.6}$$

where

$$\Omega = \Omega(k) = (1 + 4i\gamma Ck/\alpha^2)^{1/2}. \tag{6.7}$$

Performing the inverse Fourier transform of (6.2) and utilising (6.4) we can write

$$P(z, t) = \int_{-\infty}^{\infty} dz' \, \mathbb{F}(z - z', t) P(z', 0) \tag{6.8}$$

where

$$\mathbb{F}(z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, e^{-ikz} \hat{\mathbb{F}}(k, t). \tag{6.9}$$

The solution of equation (5.5) can be obtained from (6.8) and has the form (6.1) with the propagator

$$\Pi(x, t|x', 0) = \frac{1}{2\pi x} \int_{-\infty}^{\infty} dk \, \hat{\mathbb{F}}(k, t) \exp\left(-ik \ln \frac{x}{x'}\right). \tag{6.10}$$

One can construct the propagator $\Pi(x, t|x', s)$ for arbitrary times t and s ($t \geqslant s \geqslant 0$). It is determined by the equation

$$p(x, t) = \int_{0}^{\infty} dx' \, \Pi(x, t|x', s) p(x', s). \tag{6.11}$$

Taking into account (6.1), from (6.11) we obtain the integral equation for unknown $\Pi(x, t|x', s)$ of the form

$$\int_{0}^{\infty} dx' \, \Pi(x, t|x', s) \Pi(x', s|x'', 0) = \Pi(x, t|x'', 0). \tag{6.12}$$

The solution of this equation has the form

$$\Pi(x, t|x', s) = \frac{1}{2\pi x} \int_{-\infty}^{\infty} dk \frac{\hat{\mathbb{F}}(k, t)}{\hat{\mathbb{F}}(k, s)} \exp\left(-ik \ln \frac{x}{x'}\right). \tag{6.13}$$

From the construction of our propagators it follows that they obey the semigroup property

$$\Pi(x, t|x', s) = \int_{0}^{\infty} dx'' \, \Pi(x, t|x'', u) \Pi(x'', u|x', s) \qquad t \geqslant u \geqslant s \geqslant 0. \tag{6.14}$$

Equation (6.14) is similar to the Chapman–Kolmogorov equation for the conditional probability function of Markovian processes. At this stage we should stress that we cannot decide whether the propagators (6.13) have the significance of a conditional probability of the process x_t in (2.1). Only $\Pi(x, t|x', 0)$ coincides with the conditional probability because the system has no memory for previous times $t \leqslant 0$. This is a consequence of the fact that from the evolution equation like (5.5) for a single-event probability density $p(x, t)$ one cannot deduce whether the process is Markovian or not Markovian (for details, see [18]).

Equation (4.2) is a differential form of (6.11). If (6.11) holds then the time derivative of $p(x, t)$ can always be presented in the form (4.2). The propagators (evolution operators) Π fulfil the same equation as (4.2), namely

$$\frac{\partial \Pi(x, t|x', s)}{\partial t} = \int_{-\infty}^{\infty} dx'' \, \mathscr{W}_t(x|x'') \Pi(x'', t|x', s). \tag{6.15}$$

If we present $\mathscr{W}_t(x|x')$ in the form (cf (4.3))

$$\mathscr{W}_t(x|x') = \mathbb{W}_t(x|x')\theta(x') \tag{6.16}$$

then $\mathbb{W}_t(x|x')$ is defined by

$$\mathbb{W}_t(x|x') = \lim_{\varepsilon\to 0} \varepsilon^{-1}[\Pi(x, t+\varepsilon|x', t) - \delta(x-x')]. \tag{6.17}$$

It is in fact the definition of the generator of the evolution operator Π. Now, it can be easily shown that from (6.14), (6.16) and (6.17) it follows that (6.15) is fulfilled.

7. Main characteristics of the process x_t

For any state functions $F_1(x)$ and $F_2(x)$ one can determine the initial correlation function [18]

$$\langle F_1(x_t)F_2(x_0)\rangle = \int_0^\infty \mathrm{d}x_1 \int_0^\infty \mathrm{d}x_2\, F_1(x_1)F_2(x_2)\Pi(x_1, t|x_2, 0)p(x_2, 0). \tag{7.1}$$

For the special case when $F_1(x) = x^n$ and $F_2(x) = x^m$ (n, m are any natural numbers), from (7.1) we obtain

$$C_{nm}(t, 0) = \langle x_t^n x_0^m\rangle = \hat{\mathbb{F}}(-\mathrm{i}n, t)\langle x_0^{n+m}\rangle. \tag{7.2}$$

From (7.2) we can obtain the mean value [7, 8]

$$C_{10}(t, 0) = \langle x_t\rangle = \hat{\mathbb{F}}(-\mathrm{i}, t)\langle x_0\rangle \tag{7.3}$$

and fluctuations

$$\langle x_t^2\rangle - \langle x_t\rangle^2 = \hat{\mathbb{F}}(-2\mathrm{i}, t)\langle x_0^2\rangle - \hat{\mathbb{F}}^2(-\mathrm{i}, t)\langle x_0\rangle^2 \tag{7.4}$$

of the process x_t. The moments are

$$C_{n0}(t, 0) = \langle x_t^n\rangle = \hat{\mathbb{F}}(-\mathrm{i}n, t)\langle x_0^n\rangle. \tag{7.5}$$

Comparing (7.2) and (7.5) we get

$$\langle x_t^n x_0^m\rangle = \langle x_t^n\rangle \frac{\langle x_0^{n+m}\rangle}{\langle x_0^n\rangle}. \tag{7.6}$$

If the initial distribution $p(x, 0)$ is given by (2.7) then we find the decorrelation property

$$\langle x_t^n x_0^m\rangle = \langle x_t^n\rangle\langle x_0^m\rangle. \tag{7.7}$$

Let us notice that the moments $C_{n0}(t, 0)$ of x_t can be obtained directly from the characteristic function $C^z(k, t)$ of z_t. Indeed,

$$C^z(k, t) = \langle \exp(\mathrm{i}kz_t)\rangle. \tag{7.8}$$

Using (2.9) and setting $k = -\mathrm{i}n$, we observe that

$$C^z(-\mathrm{i}n, t) = \langle x_t^n\rangle. \tag{7.9}$$

By virtue of (6.3) and (7.9) we obtain the relaxation equations for the moments of x_t,

$$\frac{\mathrm{d}\langle x_t^n\rangle}{\mathrm{d}t} = -n[A + \hat{G}(-\mathrm{i}n, t)]\langle x_t^n\rangle. \tag{7.10}$$

One can check that the coefficients $M_n(x, t)$ given by (5.7) can be expressed by

$$M_1(x, t) = \left.\frac{\mathrm{d}\langle x_t\rangle}{\mathrm{d}t}\right|_{\langle x_t\rangle = x} \tag{7.11}$$

for $n = 1$ and

$$M_n(x, t) = \frac{\mathrm{d}}{\mathrm{d}t}\langle (x_t - \langle x_t\rangle)^n\rangle|_{\langle x_t^l\rangle = x^l} \qquad l = 1, 2, 3, \ldots \tag{7.12}$$

for $n = 2, 3, \ldots$.

8. The approximation methods revisited

Our exact results allow us to revise some approximation procedures [6, 11]. The most popular Fokker-Planck approximation can be obtained directly from equation (5.5) by an ordinary truncation of the Kramers-Moyal operator (5.6) and keeping only $M_1(x, t)$ and $M_2(x, t)$. It would be correct procedure if one could show that all $K_n(t)$ or rather $K_n(t)/(n-1)!$ ($n = 3, 4, \ldots$) tend to zero in some limiting cases. There are two parameters γ and α which characterise the noise y_t. We can define the intensity ε and correlation time τ of the noise by

$$\varepsilon = \gamma/\alpha^2 \qquad \tau = 1/\alpha. \tag{8.1}$$

It is argued that the Fokker-Planck approximation is correct if three limits, the long-time limit, $t \gg \tau$, the limits of small intensity and small correlation time of the noise, are carried out.

Because of (5.8) we should first consider the functions $\hat{G}(-\mathrm{i}n, t)$ ($n = 1, 2, 3, \ldots$). Let $C > 0$. Then in the long-time limit $\hat{G}(-\mathrm{i}n, t)$ do not depend on time and have the form

$$\hat{G}(-\mathrm{i}n) = -B^2 P_n(\varepsilon) + CB^2 P_n^2(\varepsilon) + C\alpha R_n(\varepsilon) \tag{8.2}$$

where

$$P_n(\varepsilon) = \frac{\varepsilon n[1 + (1 + 4C\varepsilon n)^{1/2}]}{(1 + 4C\varepsilon n)^{1/2}[1 + 2C\varepsilon n + (1 + 4C\varepsilon n)^{1/2}]} \tag{8.3}$$

$$R_n(\varepsilon) = \frac{\varepsilon[1 + (1 + 4C\varepsilon n)^{1/2}]}{1 + 2C\varepsilon n + (1 + 4C\varepsilon n)^{1/2}}. \tag{8.4}$$

One can check that

$$\lim_{n\to\infty} \hat{G}(-\mathrm{i}n) = -B^2/4C. \tag{8.5}$$

In the long-time limit the function $K_n(t)$ (5.8) can be presented in the form

$$K_1 = \hat{G}(-\mathrm{i})$$
$$K_n = \hat{G}(-\mathrm{i}n) - \sum_{l=1}^{n-1} \binom{n-1}{l} K_{n-l} \qquad n = 2, 3, \ldots. \tag{8.6}$$

Let ε be sufficiently small. From (8.2)-(8.6) it is seen that a systematic power series expansion of K_n as a function of ε is impossible for sufficiently large n. But in (5.6) we need rather $K_n/(n-1)!$. One can show that

$$\lim_{\varepsilon\to 0} K_n/(n-1)! = 0 \qquad \lim_{n\to\infty} K_n/(n-1)! = 0. \tag{8.7}$$

In these circumstances we can expand $P_n(\varepsilon)$ and $R_n(\varepsilon)$ for small ε and not too large $n=1,2,\ldots,N$ ($4C\varepsilon N \ll 1$) and neglect all $K_n/(n-1)!$ for $n=N+1, N+2, \ldots$. Then, to order ε^2, the generator (5.6) becomes

$$\mathscr{L}=(A-B^2\varepsilon+C\alpha\varepsilon-C^2\alpha\varepsilon^2+4CB^2\varepsilon^2)\frac{\partial}{\partial x}x + (B^2\varepsilon+C^2\alpha\varepsilon^2-12CB^2\varepsilon^2)\frac{\partial^2}{\partial x^2}x^2+4CB^2\varepsilon^2\frac{\partial^3}{\partial x^3}x^3 \tag{8.8}$$

and $K_4=\mathrm{O}(\varepsilon^3)$, $K_5=\mathrm{O}(\varepsilon^4)$, and so on.

The usual Fokker-Planck generator $\mathscr{L}_{\mathrm{FP}}$ can be obtained from (8.8) by keeping the terms of order ε. If α is large (small correlation time τ), $\alpha\sim 1/\varepsilon$, then

$$\mathscr{L}_{\mathrm{FP}}=(A+C\alpha\varepsilon-B^2\varepsilon-C^2\alpha\varepsilon^2)\frac{\partial}{\partial x}x+(B^2\varepsilon+C^2\alpha\varepsilon^2)\frac{\partial^2}{\partial x^2}x^2. \tag{8.9}$$

We should compare our equations (8.8) and (8.9) with those obtained in [6,11]. Our ε (8.1) corresponds to D in [6] and Γ in [11]. Using equations (3.13)-(3.16) in [6] we obtain the same equation as (8.9). In [11], Wódkiewicz derived the generator $\mathscr{L}$ up to order ε^2 (cf his equation (4.14a)). This corresponds to our equation (8.8). One can check that his equation (4.14a) leads to a quite different generator than (8.8). His equation (4.13) is incorrect. It can be easily seen assuming that the parameter C in (2.1) is equal to zero. Then from (8.2) we get

$$\hat{G}(-\mathrm{i}n)=-B^2\varepsilon n \tag{8.10}$$

and by virtue of (5.8) we have

$$K_{n+1}=(-1)^{n+1}B^2\varepsilon\sum_{l=0}^{n}(-1)^l\binom{n}{l}(l+1). \tag{8.11}$$

It is seen that $K_n=0$ for $n=3,4,5,\ldots$. For the model (2.1) with the linear noise y_t we obtain the exact Fokker-Planck generator. From equation (4.13) in [11] one obtains terms proportional to $\partial^3/\partial x^3$ and $\partial^4/\partial x^4$ which are different from zero.

Finally, let us consider the following equation:

$$\dot{x}=-\mu^2(A+\gamma C/\alpha)x-\mu(By+Cy^2-\gamma C/\alpha)x. \tag{8.12}$$

If $\mu=1$ then (8.12) reduces to (2.1). Stratonovich [19] proved that non-Markovian processes described by equations of type (8.12) may be approximated by diffusion Markovian processes in the limit $\mu\to 0$. More precisely, for fixed $t'=\mu^2 t$, let $x_t=x(t'/\mu^2)=\tilde{x}_\mu(t')$. Then

$$\lim_{\mu\to 0}\tilde{x}_\mu(t')=\tilde{x}_0(t') \tag{8.13}$$

where $\tilde{x}_0$ is a diffusion Markovian process. Using theorem 1.1 of [19], one can obtain the generator of $\tilde{x}_0$. It is interesting that this generator for the process $\tilde{x}_0$ obtained from (8.12) has exactly the form (8.9).

9. Summary

The model (2.1) we have considered represents the simplest exactly solvable stochastic model with non-linear noise. It describes many relaxation phenomena and generates the non-Markovian process x_t. We have presented the elimination procedure of the additional degree of freedom y from the two-dimensional stochastic process (x_t, y_t). We have derived fundamental formulae which characterise the process x_t. All these formulae can be derived by using other methods but we think that the method presented is non-standard, more elegant and interesting.

It should be pointed out that the results obtained are valid for any assumed values of the parameters of the model. Therefore, we have no problem of the domain of validity of results as in approximative theories (small or large values of some parameters, the adiabatic elimination scheme, the decoupling theory and so on, cf [20]).

Acknowledgment

I thank J Sładkowski for helpful discussions and for reading the manuscript, and the referees for their remarks. This work was supported in part by the research programme CPBP 1.1.4.

Appendix

We want to present the exponential operator on the left-hand side of equation (5.4) in the anti-normal order

$$\exp\left(-zx\frac{\partial}{\partial x}\right) = \sum_{n=0}^{\infty} \beta_n(z)\frac{\partial^n}{\partial x^n}x^n. \tag{A1}$$

Differentiation with respect to the parameter z of both sides of (A1) and using the commutation relation

$$\left[\frac{\partial^n}{\partial x^n}, x\right] = n\frac{\partial^{n-1}}{\partial x^{n-1}} \tag{A2}$$

leads to the equation for the coefficients $\beta_n(z)$,

$$\frac{\mathrm{d}\beta_n(z)}{\mathrm{d}z} = (n+1)\beta_n(z) - \beta_{n-1}(z) \tag{A3}$$

where $n = 0, 1, 2, \ldots$, and $\beta_{-1}(z) = 0$. The solution of this equation has the form

$$\beta_n(z) = \frac{(-1)^n}{n!}\mathrm{e}^z(\mathrm{e}^z - 1)^n \tag{A4}$$

which leads to the identity (5.4).

References

[1] Haken H 1978 *Synergetics—An Introduction* (Berlin: Springer)
[2] Nicolis G and Prigogine I 1977 *Self-Organization in Nonequilibrium Systems* (New York: Wiley)
[3] Goel N W and Richter-Dyn N 1974 *Stochastic Models in Biology* (New York: Academic)
[4] Stratonovich R L 1967 *Topics in the Theory of Random Noise* (New York: Gordon and Breach)
Ning-Mao Xia and Boyce W E 1984 *SIAM J. Appl. Math.* **44** 1192
[5] van Kampen N G 1976 *Phys. Rep.* **24C** 171

[6] San Miguel M and Sancho J M 1981 *Z. Phys.* B **43** 361

[7] Wódkiewicz K and Zubairy M S 1983 *J. Math. Phys.* **24** 1401
Becker W, Scully M O, Wódkiewicz K and Zubairy M S 1984 *Phys. Rev.* A **30** 2245

[8] van Hemmen J L and Rzążewski K 1983 *Phys. Rev.* A **28** 474

[9] Gardiner C W 1985 *Handbook of Stochastic Methods* (Berlin: Springer)

[10] Łuczka J 1987 *J. Stat. Phys.* **47** 505

[11] Wódkiewicz K 1982 *J. Math. Phys.* **23** 2179

[12] Martin P C, Siggia E D and Rose H A 1973 *Phys. Rev.* A **8** 423

[13] van Kampen N G 1985 *Phys. Rep.* **124C** 69

[14] Kramers H A 1940 *Physica* **7** 248
Moyal J E 1949 *J. R. Stat. Soc.* B **11** 150

[15] Leontiev A F 1975 *Results of Sciences and Technics* (*Mathematical Analysis* **13**) (Moscow: Viniti) p 5

[16] van Kampen N G 1980 *Phys. Lett.* **76A** 104

[17] Prudnikov A P, Brychkov Yu A and Marichev O I 1981 *Integrals and Series* vol 1 (Moscow: Nauka) § 1.4.7

[18] Hänggi P and Thomas H 1977 *Z. Phys.* B **26** 85
Hänggi P, Thomas H, Grabert H and Talkner P 1978 *J. Stat. Phys.* **18** 155

[19] Stratonovich R L 1968 *Conditional Markov Processes and their Application to Optimal Control* (Amsterdam: Elsevier)

[20] Jung P and Hänggi P 1987 *Phys. Rev.* A **35** 4464
Horsthemke W, Doering C R, Lefever R and Chi A S 1985 *Phys. Rev.* A **31** 1123
Fox R F 1986 *Phys. Rev.* A **34** 4525
Masoliver J, West B J and Linderberg K 1987 *Phys. Rev.* A **35** 3086
Faetti S and Grigolini P 1987 *Phys. Rev.* A **36** 441